

ON MINIMAL SINGULAR METRICS OF LINE BUNDLES WHOSE STABLE BASE LOCI ADMIT HOLOMORPHIC TUBULAR NEIGHBORHOODS

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ABSTRACT. We investigate the singularity of a minimal singular metric on a big line bundle L over a projective manifold when the stable base locus Y of L is a submanifold of codimension $r \geq 1$. Under some assumptions on the normal bundle and a neighborhood of Y , we give a concrete description of the singularity of a minimal singular metric. We apply this result to a higher (co-)dimensional analogue of Zariski's example, in which the line bundle L is not semi-ample, however it is nef and big.

1. INTRODUCTION

Let X be a projective manifold and L be a big line bundle on X . Our interest is in *minimal singular metrics* of L . Minimal singular metrics of L are metrics of L with the mildest singularities among singular Hermitian metrics of L whose local weights are plurisubharmonic ([DPS, Definition 1.4], see also §2 here). Minimal singular metrics always exist when L is pseudo-effective [DPS, Theorem 1.5]. Indeed, in this case, for any C^∞ Hermitian metric h on a pseudo-effective line bundle L , the *equilibrium metric*

$$h_e := h \cdot \exp(-\sup\{\varphi : X \rightarrow \mathbb{R} \cup \{-\infty\} \mid \varphi : \Theta_h\text{-plurisubharmonic and } \varphi \leq 0\})$$

is a minimal singular metric (Θ_h is the curvature tensor of h , see also §2 for details).

The goal of this paper is to describe the singularity of a minimal singular metric explicitly. Denote by $Y = \text{SB}(L)$ the stable base locus of L . It is easily observed that a minimal singular metric is locally bounded on each point in $X \setminus Y$ (see Example 2.4). Thus we are interested in the singularity of a minimal singular metric along Y . In the present paper, we investigate this question under the following conditions: (i) Y is a smooth (i.e. non-singular) compact subvariety of codimension r , and (ii) the normal bundle $N_{Y/X}$ admits a direct decomposition $N_{Y/X} = N_1 \oplus N_2 \oplus \cdots \oplus N_r$ into r negative line bundles.

One of the most important examples of (X, L, Y) with (i) and (ii) is Nakayama's example [N, IV, §2.6], which admits no Zariski decomposition even after modifications. In this case, the second author showed that a minimal singular metric can be concretely constructed by using the convex set

$$\square_L = \{\alpha = (\alpha_1, \alpha_2, \dots, \alpha_r) \in \mathbb{R}_{\geq 0}^r \mid |\alpha| \leq 1, c_1(L|_Y) + \sum_{\lambda=1}^r \alpha_\lambda c_1(N_\lambda^{-1}) : \text{pseudo-effective}\},$$

where $|\alpha| := \sum_{\lambda=1}^r \alpha_\lambda$ [K1]. Note that Y is an abelian surface in this Nakayama's example. One main application of our main result is the following:

THEOREM 1.1. *Let X be a projective manifold, L be a big line bundle on X , and $Y = \text{SB}(L)$ be the stable base locus of L . Assume the conditions (i) and (ii). Assume*

also that Y is an abelian variety, $L|_Y \otimes N_\lambda^{-1}$ is positive for each $\lambda = 1, 2, \dots, r$, and that $N_\lambda \cong N_\mu$ for each λ and μ . Then the local weight function $\varphi_{\min,L}$ of a minimal singular metric $h_{\min,L}$ (i.e. $\varphi_{\min,L}$ is a locally defined function such that $h_{\min,L} = e^{-\varphi_{\min,L}}$) can be written as

$$\varphi_{\min,L}(z, y) = \log \max_{\alpha \in \square_L} \prod_{\lambda=1}^r |z_\lambda|^{2\alpha_\lambda} + O(1)$$

on a neighborhood of each point of Y , where y is a coordinate of Y and $z = (z_1, z_2, \dots, z_r)$ is a system of local defining functions of Y (Here we are formally regarding 0^0 as 1).

Our main result is the following:

THEOREM 1.2. *Let X be a projective manifold, L be a big line bundle on X , and $Y = SB(L)$ be the stable base locus of L . Assume three conditions (i), (ii), and (iii) Y admits a holomorphic tubular neighborhood in X : i.e. there exists a neighborhood V of Y in X and a neighborhood \tilde{V} of the zero section in $N_{Y/X}$ such that there exists a biholomorphism $i: V \rightarrow \tilde{V}$ such that $i|_Y$ coincides with the natural isomorphism. Assume also that $L|_Y \otimes N_\lambda^{-1}$ and $K_Y^{-1} \otimes N_\lambda^{-1}$ are positive for each $\lambda = 1, 2, \dots, r$. Take a C^∞ Hermitian metric $h_{L|_Y}$ on $L|_Y$ and h_{N_λ} on N_λ for each λ with $\Theta_{h_{L|_Y} \otimes h_{N_\lambda}^{-1}} > 0$. Then the local weight function $\varphi_{\min,L}$ of a minimal singular metric $h_{\min,L}$ can be written as*

$$\varphi_{\min,L}(z, y) = \log \max_{\alpha \in \square_L} \left(\prod_{\lambda=1}^r |z_\lambda|^{2\alpha_\lambda} \right) \cdot e^{(\varphi_\alpha)_e(y)} + O(1)$$

on a neighborhood of each point of Y , where y is a coordinate of Y , $z = (z_1, z_2, \dots, z_r)$ is a system of local defining functions of Y , and $(\varphi_\alpha)_e$ is the local weight function of the equilibrium metric of $h_{L|_Y} \otimes h_{N_1}^{-\alpha_1} \otimes h_{N_2}^{-\alpha_2} \cdots \otimes h_{N_r}^{-\alpha_r}$ (see §2 for the notion of the “metric” $h_{L|_Y} \otimes h_{N_1}^{-\alpha_1} \otimes h_{N_2}^{-\alpha_2} \cdots \otimes h_{N_r}^{-\alpha_r}$ for real α_λ ’s).

Theorem 1.2 can be regarded as a higher codimensional generalization of both the main results in [K1] and [K2] (In [K2], Theorem 1.2 is shown in the case $r = 1$). Note that $0 \in \square_L$ if $L|_Y$ is pseudo-effective. Thus, in this case, it follows from Theorem 1.2 that $h_{\min,L}|_Y \leq (h_{L|_Y})_e \cdot e^{O(1)}$, which means that $h_{\min,L}|_Y$ has the mildest singularities. Therefore we have the following:

COROLLARY 1.3. *Let X, L , and Y be as in Theorem 1.2. Assume that $L|_Y$ is pseudo-effective. Then $h_{\min,L}|_Y$ is a minimal singular metric of $L|_Y$.*

The assumptions (i), (ii) and (iii) are essential and can not be dropped, see §6.3. We can apply these results to, for example, the following higher (co-)dimensional analogue of Zariski’s example (see §6.2).

COROLLARY 1.4. *Take two general quadric surfaces Q_1 and Q_2 in \mathbb{P}^3 and fix general N points p_1, p_2, \dots, p_N in $Q_1 \cap Q_2$ ($N \geq 12$). Denote by $\pi: X := \text{Bl}_{\{p_1, p_2, \dots, p_N\}} \mathbb{P}^3 \rightarrow \mathbb{P}^3$ the blow-up of \mathbb{P}^3 at these N points, and by D_1 and D_2 the strict transform of Q_1 and Q_2 , respectively. Then the local weight function $\varphi_{\min,L}$ of a minimal singular metric of $L := \pi^* \mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathcal{O}_X(D_1)$ can be written as*

$$\varphi_{\min,L}(z, y) = \frac{N-12}{N-8} \cdot \log(|z_1|^2 + |z_2|^2) + O(1)$$

on a neighborhood of each point of $Y := D_1 \cap D_2$, where y and $z = (z_1, z_2)$ are as in Theorem 1.1. Especially, when $N = 12$, L is nef big and not semi-ample, and admits a C^∞ Hermitian metric with semi-positive curvature.

The proof of Theorem 1.2 is based on [K2]. We first study a special case where X is a projective space bundle over Y and L is a relative tautological bundle. After that, we apply the concrete description of a minimal singular metric for this spacial case to the study of general (X, L, Y) by using maximal construction technique (here we use the assumption (iii)). We also study the condition (iii) itself by using Grauert's theory [G]. We give a sufficient condition for (iii) when, for example, Y has a trivial tangent bundle. As an application, we can deduce Theorem 1.1 from Theorem 1.2.

The organization of the paper is as follows. In §2, we collect some fundamental notations and facts on projective bundles and singular Hermitian metrics. In §3, we show the main result in the special case in which X is the total space of a projective bundle. In §4, we prove Theorem 1.2 in the general configuration. In §5, we give a sufficient condition for (iii) by using Grauert's theory. Here we also show Theorem 1.1. In §6, we give some examples.

2. PRELIMINARIES

2.1. Notations on projective bundles. Let Y be a compact complex manifold and M_1, M_2, \dots, M_{r+1} be holomorphic line bundles on Y . Let $\{U_j\}_j$ be an open cover of Y . Assume that each U_j is sufficiently small so that $M_\lambda|_{U_j}$ is trivial for every λ and j . Then there exist local holomorphic trivializing sections $s_{j,\lambda}$ of M_λ on U_j . Let E be the vector bundle $M_1 \oplus M_2 \oplus \dots \oplus M_{r+1}$. Then $(s_{j,\lambda})_\lambda$ forms a holomorphic frame of E on U_j . Let $(\xi_{j,\lambda})_\lambda$ be the dual frame of $(s_{j,\lambda})_\lambda$.

We fix the notations on $\mathbb{P}(E)$ as follows. Let us denote by $\mathbb{P}(E)$ the projective bundle of hyperplanes of E over Y , i.e. $\mathbb{P}(E) := \bigcup_y (E_y^* \setminus 0)/\mathbb{C}^*$. On the other hand, we will denote the bundle of lines by $\mathbf{P}(E)$ in this paper. Let π denote the natural projection $\mathbb{P}(E) \rightarrow Y$. We will use the notation $\mathbb{P}(E)|_{U_j}$ to denote $\pi^{-1}(U_j)$. A homogeneous coordinate $([x_{j,1} : x_{j,2} : \dots : x_{j,r+1}], y)$ on $\mathbb{P}(E)|_{U_j}$ is defined as $[x_{j,1}\xi_{j,1} + x_{j,2}\xi_{j,2} + \dots + x_{j,r+1}\xi_{j,r+1}] \in \mathbb{P}(E)_y$. Here $\mathbb{P}(E)_y$ denotes the fiber $\pi^{-1}(y)$. Let $U_j^{(\lambda)}$ be an open set $\{([x_{j,1}\xi_{j,1} + x_{j,2}\xi_{j,2} + \dots + x_{j,r+1}\xi_{j,r+1}], y) \mid y \in U_j, x_{j,\lambda} \neq 0\}$ of $\mathbb{P}(E)$. Note that $\{U_j^{(\lambda)}\}_{j,\lambda}$ forms an open cover of $\mathbb{P}(E)$. The tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$ is defined by setting its fiber on $([\xi], y)$ as $E_y/\text{Ker } \xi$, where ξ denotes an element of $E_y^* \setminus 0$. Let Γ_λ be the divisor of $\mathbb{P}(E)$ defined as $\mathbb{P}(M_1 \oplus M_2 \oplus \dots \widehat{M_\lambda} \oplus \dots \oplus M_{r+1})$. The following fact is obtained by a simple computation.

LEMMA 2.1. (i) $[\Gamma_\lambda] \otimes \pi^* M_\lambda = \mathcal{O}_{\mathbb{P}(E)}(1)$, where $[\Gamma_\lambda]$ denotes the line bundle defined by the divisor Γ_λ .
(ii) $N_{Y/X} = \bigoplus_{\lambda=1}^r \mathcal{O}_{\mathbb{P}(E)}(1)|_Y \otimes M_\lambda^{-1}$.

2.2. Singular Hermitian metrics. In this subsection, we review some properties of singular Hermitian metrics on line bundles.

DEFINITION 2.2. Let X be a (possibly non-compact) complex manifold and L be a line bundle on X . A *singular Hermitian metric* h on L is defined as a fiber metric of L such that, for each trivialization $L|_U \cong U \times \mathbb{C}$, h has the form $\|s\|_h^2 = |s|^2 e^{-\phi}$ on U , where $\phi \in L_{\text{loc}}^1(U)$. In this situation, we will write as $h = e^{-\phi}$ and call ϕ as a *local weight*. Note that ϕ is a collection of a function defined on each small open set. The *curvature* of a singular Hermitian metric $h = e^{-\phi}$ is defined as a $(1,1)$ -current $\Theta_h = \sqrt{-1}\partial\bar{\partial}\phi$. Note that the curvature of h is a globally defined current on X .

A singular Hermitian metric $h = e^{-\phi}$ is *semi-positively curved* (or h admits *semi-positive curvature*) if its local weight ϕ is psh on each locus. In this case, its curvature is non-negative as a $(1,1)$ -current.

Let h_1 and h_2 be singular Hermitian metrics on L . We say that h_1 is *more singular* than h_2 when, for each relatively compact set U , there is a constant $C > 0$ such that the inequality $h_1 \geq Ch_2$ holds on U . In this case we write $h_1 \gtrsim_{\text{sing}} h_2$. We say that h_1 and h_2 have *equivalent singularities* (written $h_1 \sim_{\text{sing}} h_2$) when both $h_1 \lesssim_{\text{sing}} h_2$ and $h_1 \gtrsim_{\text{sing}} h_2$ hold. A semi-positively curved singular Hermitian metric h on L is *minimal singular* if $h \lesssim_{\text{sing}} h'$ for any semi-positively curved singular Hermitian metric h' . If X is compact, the constant in the definition of $h \gtrsim_{\text{sing}} h'$ can be chosen globally, i.e. there exists a constant C such that $h_1 \geq Ch_2$ on X .

To investigate singular Hermitian metrics, it will be convenient to consider globally defined functions corresponding to their local weights. Therefore we introduce the notion of θ -psh functions here. Let θ be a smooth real $(1,1)$ -form. We say that a function $u \in L_{\text{loc}}^1(X)$ is a θ -psh function when the inequality $\theta + i\partial\bar{\partial}u \geq 0$ holds as currents. We denote the set of θ -psh functions on X by $\text{PSH}(X, \theta)$.

Let L be a holomorphic line bundle on X . Fix a smooth Hermitian metric h_0 on L with curvature θ . Then there is a one-to-one correspondence between θ -psh functions u and semi-positively curved singular Hermitian metrics $h_0 \cdot e^{-u}$ on L . We define a minimal singular θ -psh similarly to the case of a minimal singular metric. Namely, a θ -psh function u is a minimal singular θ -psh function if, for every θ -psh function u' , there exists a (local) constant C such that $u \geq u' + C$ on each compact set. For an \mathbb{R} -line bundle L (i.e. a formal “line bundle” corresponding to an \mathbb{R} -divisor), a notion of singular Hermitian metric on L is well-defined formally in this sense.

EXAMPLE 2.3. Assume X is compact and L is *pseudo-effective*, i.e. L admits a semi-positively curved singular Hermitian metric. Fix a smooth metric h with curvature θ . We can construct a θ -psh function V_θ by

$$V_\theta := \sup\{v \in \text{PSH}(X, \theta) \mid v \leq 0\}.$$

It is easily observed that V_θ is a minimal singular θ -psh function. The corresponding singular Hermitian metric $h \cdot e^{-V_\theta}$ is denoted by h_e , which is called the *equilibrium metric*.

EXAMPLE 2.4. Fix a smooth Hermitian metric h_0 on L . Let $f_1, f_2, \dots, f_N \in H^0(X, L)$ be global holomorphic sections of L . Then we can define a singular Hermitian metric h by the formula

$$\|f\|_h^2 := \frac{\|f\|_{h_0}^2}{\sum_{j=1}^N \|f_j\|_{h_0}^2}.$$

In this manner, we obtain a semi-positively curved singular Hermitian metric h which is smooth on the Zariski open set $\bigcup_{j=1}^N \{f_j \neq 0\}$. If L is big, we can choose finite number of sections $f_1, \dots, f_N \in H^0(X, L^m)$ for sufficiently large m such that $\{f_1 = \dots = f_N = 0\} = \text{SB}(L)$ ([Laz, 2.1.21]). Here we denote the tensor product $L^{\otimes m}$ by L^m . Then, we have a singular Hermitian metric on L^m which is smooth on $X \setminus \text{SB}(L)$. By taking m -th root, we can define a singular Hermitian metric on L (we call it a *Bergman-type metric* on L obtained by f_1, \dots, f_N).

EXAMPLE 2.5. Let X be a compact complex manifold and L be a line bundle. Fix a smooth volume form dV on X and a smooth metric $h = e^{-\phi}$ on L . Let θ be the curvature of h . We define a θ -psh function $V_{\phi, B}$ by

$$V_{\phi, B} := V_{h, B} := \sup \left\{ \frac{1}{m} \log |f|_{h^m}^2 \mid m \in \mathbb{Z}, f \in H^0(X, L^m), \int_X |f|_{h^m}^2 dV \leq 1 \right\}.$$

The corresponding singular Hermitian metric on L and its local weight are denoted by $h_B = e^{-\phi_B}$. By Proposition 2.6, h_B is a minimal singular metric when L is big.

We can use this construction when L is a \mathbb{Q} -line bundle with a smooth “metric” $h = e^{-\phi}$, that is, for some integer $m > 0$, L^m is an ordinary line bundle and $h^m = e^{-m\phi}$ is a smooth metric on L^m . In this case, take the smallest integer $m > 0$ such that L^m is a \mathbb{Z} -line bundle. Then we define ϕ_B by $(1/m)(m\phi)_B$.

To compare the metrics h_e and h_B , we need the following proposition.

PROPOSITION 2.6 ([K2, Lemma 2.10]). *Let X be a projective manifold and L be a big line bundle. Let $h = e^{-\phi}$ be a smooth Hermitian metric on L . Fix a smooth volume form dV on X . Then, there is a constant C such that the inequality*

$$V_{\phi, B} - C \leq V_{\theta} \leq V_{\phi, B}$$

holds.

Before starting the proof, we shall explain how we use Proposition 2.6 in §3. We will apply it to a family of \mathbb{Q} -line bundles of the form

$$L^\alpha := L_1^{\alpha_1} \otimes L_2^{\alpha_2} \otimes \dots \otimes L_{r+1}^{\alpha_{r+1}},$$

where L_λ are \mathbb{Z} -line bundles, $\alpha_\lambda \geq 0$ and $\alpha_1 + \dots + \alpha_{r+1} = 1$. Let $e^{-\phi_\lambda}$ be a fixed smooth metric on L_λ and m be the smallest positive integer such that $(L^\alpha)^m$ is a \mathbb{Z} -line bundle. Then, the local weight $m\phi_\alpha := m \sum_\lambda \alpha_\lambda \phi_\lambda$ defines a smooth metric on $(L^\alpha)^m$. The constant C in Proposition 2.6 depends only on C_1 , C_2 and $C(\phi)$, which will be defined in the proof below. We note that C_1 and C_2 are independent of the choice of line bundles, and $C(\phi)$ only depends on the differences $(\sup_{B_j''} - \inf_{B_j''})\phi$ concerning local weights, where B_j'' denotes a small open ball such that $\{B_j''\}$ is a covering of X . Thus there exists a constant C_3 depending on the metrics $e^{-\phi_\lambda}$ and independent of α , such that

$$V_{m\phi^\alpha, B} - \log(C_1 C_2) - mC_3 \leq V_{m\theta_\alpha} \leq V_{m\phi^\alpha, B}.$$

Dividing by m , we have that

$$V_{\phi^\alpha, B} - \frac{1}{m} \log(C_1 C_2) - C_3 \leq V_{\theta_\alpha} \leq V_{\phi^\alpha, B}.$$

In conclusion, there exists a constant C such that we have

$$(1) \quad V_{\phi_\alpha, B} - C \leq V_{\theta_\alpha} \leq V_{\phi_\alpha, B}$$

for every α such that $\alpha_\lambda \geq 0$ and $\alpha_1 + \alpha_2 \cdots + \alpha_{r+1} = 1$.

Proof. First we prove the inequality $V_\theta \leq V_{\phi, B}$. Since L is big, there exists a singular Hermitian metric ψ_+ on L such that its curvature is a Kähler current, i.e. $\Theta_{\psi_+} \geq \epsilon\omega$ for some $\epsilon > 0$ and some Kähler form ω . Define a θ -psh function V_+ by $\psi_+ = \phi + V_+$. We can assume that $V_+ \leq 0$. Let $V_\ell := (1 - 1/\ell)V_\theta + (1/\ell)V_+$ and $\phi_\ell := \phi + V_\ell$. Then the curvature of the metric $e^{-\phi_\ell}$ is a Kähler current. Now consider the following approximations:

$$V_{\phi, B, m} := \sup^* \left\{ \frac{1}{m} \log |f|_{m\phi}^2 : f \in H^0(X, L^m), \int_X |f|^2 e^{-m\phi} dV \leq 1 \right\},$$

$$V_{\phi_\ell, B, m} := \sup^* \left\{ \frac{1}{m} \log |f|_{m\phi_\ell}^2 : f \in H^0(X, L^m), \int_X |f|^2 e^{-m\phi_\ell} dV \leq 1 \right\}.$$

Then we have that $V_\ell + V_{\phi_\ell, B, m} \leq V_{\phi, B, m}$. Applying Demailly's approximation theorem ([D2, Theorem 13.21]) to ϕ_ℓ , we have that $\phi_{\ell, m} \geq \phi_\ell - C/m$, where C is independent of ℓ and m . Hence we have $V_\ell - \frac{C}{m} \leq V_{\phi_\ell, B, m} \leq V_{\phi, B, m}$. By letting $m \rightarrow \infty$, we obtain $V_\ell \leq V_{\phi, B}$. After that, letting $\ell \rightarrow \infty$, we have that $V_\theta \leq V_{\phi, B}$.

Next we prove the inequality $V_{\phi, B} - C \leq V_\theta$. Fix a collection of open coordinate balls $B'_j \subset B''_j \subset B_j$ such that $\{B'_j\}_j$ is an open cover of X and the radii of B'_j , B''_j and B_j are $1/2$, 1 and 2 respectively. Fix a local trivialization of L . Take $f \in H^0(mL)$ with $\int_X |f|^2 e^{-m\phi} dV \leq 1$. Then, for every $p \in B'_j$, we have that

$$\begin{aligned} |f(p)|^2 &\leq \frac{1}{\pi^n (1/2)^{2n}/n!} \int_{|z-p| < 1/2} |f|^2 d\lambda \\ &\leq C_1 C_2 \cdot e^{m \sup_{B''_j} \phi} \int_{|z-p| < 1/2} |f|^2 e^{-m\phi} dV \\ &\leq C_1 C_2 \cdot e^{m \sup_{B''_j} \phi}. \end{aligned}$$

Here we write the constants as $C_1 := \frac{1}{\pi^n (1/2)^{2n}/n!}$ and $C_2 := \sup_{B''_j} d\lambda/dV$. Thus we have that $|f(p)|^2 e^{-m\phi(p)} \leq C_1 C_2 \cdot \exp(m(\sup_{B''_j} \phi - \phi(p))) \leq C_1 C_2 \cdot \exp(m(\sup_{B''_j} - \inf_{B''_j} \phi))$. Taking logarithm and dividing by m , we have

$$\frac{1}{m} \log |f(p)|_{m\phi}^2 \leq (\log C_1 C_2)/m + \left(\sup_{B''_j} - \inf_{B''_j} \right) \phi.$$

It follows that

$$V_{\phi, B, m}(p) \leq (\log C_1 C_2)/m + \left(\sup_{B''_j} - \inf_{B''_j} \right) \phi.$$

The right-hand side is estimated by using C_1, C_2 and a constant $C(\phi)$ depends only on ϕ . Taking supremum over m , we have that $V_{\phi, B}(p) \leq \log(C_1 C_2) + C(\phi) =: C$. Considering all B_j , we have $V_{\phi, B} - C \leq V_\theta$ for some constant C . \square

3. PROJECTIVE BUNDLES

3.1. Settings in the case of \mathbb{P}^r -bundle. Let Y be a projective manifold, and M_1, M_2, \dots, M_{r+1} be line bundles. We assume that the first r line bundles M_1, \dots, M_r are ample (we do not assume the ampleness of M_{r+1}). Define a manifold X by $X := \mathbb{P}(M_1 \oplus M_2 \oplus \dots \oplus M_{r+1})$ and a line bundle L on X by $L := \mathcal{O}_{\mathbb{P}(M_1 \oplus M_2 \oplus \dots \oplus M_{r+1})}(1)$. Let us recall that $\mathbb{P}(E)$ denotes the projective bundle of *hyperplanes* of E . Let π denote the natural projection $X \rightarrow Y$. We regard Y as a submanifold of X via the inclusion induced by the projection $M_1 \oplus M_2 \oplus \dots \oplus M_{r+1} \rightarrow M_{r+1}$. Let h_λ ($1 \leq \lambda \leq r+1$) be smooth metrics on M_λ and θ_λ be the curvature forms of h_λ . Here we assume that each h_λ ($1 \leq \lambda \leq r$) has a positive curvature, i.e. θ_λ is a positive $(1,1)$ -form for each $\lambda = 1, \dots, r$. Let us denote by $h_L = e^{-\varphi_L}$ the naturally induced metric on L from h_1, \dots, h_{r+1} by considering the Euler sequence. We denote by θ_L the curvature of h_L . Let \square_L be a convex set defined in §1 as follows:

$$\square_L = \{\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{R}_{\geq 0}^r \mid |\alpha| \leq 1, c_1(L|_Y) + \sum_{\lambda=1}^r \alpha_\lambda c_1(M_\lambda \otimes L|_Y^{-1}) : \text{pseudo-effective}\},$$

where $|\alpha|$ denotes $\alpha_1 + \alpha_2 + \dots + \alpha_r$. Here we use the direct decomposition $N_{Y/X} = \bigoplus_{\lambda=1}^r (L \otimes \pi^* M_\lambda^{-1})|_Y$ (see Lemma 2.1).

We shall show the following theorem.

THEOREM 3.1. *Let Y , M_λ , X , L and h_λ be as above. For an r -tuple of real numbers $\alpha = (\alpha_1, \dots, \alpha_r)$ and a real number $\alpha_{r+1} := 1 - \alpha_1 - \dots - \alpha_r$, define a function $u_\alpha(x)$ on X as follows:*

$$u_\alpha(x) := \alpha_1 \log |s_1|_{h_1}^2 + \alpha_2 \log |s_2|_{h_2}^2 + \dots + \alpha_{r+1} \log |s_{r+1}|_{h_{r+1}}^2 + \pi^* V_{\theta_\alpha}.$$

Here, s_λ denotes the canonical section of a divisor Γ_λ , \widehat{h}_λ denotes the metric on the line bundle $[\Gamma_\lambda]$ defined by $h_L/\pi^* h_\lambda$, and θ_α denotes the $(1,1)$ -form $\sum_{\lambda=1}^r \alpha_\lambda \theta_\lambda + (1 - |\alpha|)\theta_{r+1}$. Then,

- (i) u_α is a θ_L -psh function.
- (ii) For fixed $x \in X$, the function $u_\alpha(x)$ attains a maximum value on $\alpha \in \square_L$.
- (iii) Define a function $\widehat{V}(x)$ on X by $\widehat{V}(x) := \max_{(\alpha_1, \dots, \alpha_r) \in \square_L} u_\alpha(x)$. Then \widehat{V} is upper semi-continuous (u.s.c.) and θ_L -psh.
- (iv) \widehat{V} is a minimal singular θ_L -psh function.

3.2. The relation between Theorem 3.1 and Theorem 1.2. Before proving Theorem 3.1, we shall explain the relation between Theorem 3.1 and Theorem 1.2. We assume that X , Y and L are as in Theorem 1.2. We can construct a “projective bundle model” $(\widetilde{X}, \widetilde{Y}, \widetilde{L})$, to which we can apply Theorem 3.1. We define \widetilde{X} by $\mathbb{P}(\mathbb{I}_Y \oplus N_{Y/X}^*)$, \widetilde{Y} by $\mathbb{P}(\mathbb{I}_Y)$, and \widetilde{L} by $\mathcal{O}_{\mathbb{P}(\mathbb{I}_Y \oplus N_{Y/X}^*)}(1)$. In §4, we use a trick called *maximum construction* to get a minimal singular metric on L from that on \widetilde{L} .

To check that $(\widetilde{X}, \widetilde{Y}, \widetilde{L})$ satisfies the assumption of Theorem 3.1, we have to choose appropriate line bundles M_λ on \widetilde{Y} as in the following lemma.

LEMMA 3.2. Let N_λ be line bundles as in Theorem 1.2. If we take as $M_\lambda = L|_Y \otimes N_\lambda^{-1}$ for $\lambda = 1, \dots, r$ and $M_{r+1} = L|_Y$, we have that $\tilde{X} \cong \mathbb{P}(M_1 \oplus \dots \oplus M_{r+1})$ and $\tilde{L} = \mathcal{O}_{\mathbb{P}(M_1 \oplus M_2 \oplus \dots \oplus M_{r+1})}(1)$.

Proof. By taking as above, we have that

$$\begin{aligned} \mathbb{P}(M_1 \oplus \dots \oplus M_{r+1}) &= \mathbb{P}(L|_Y \otimes (N_1^{-1} \oplus \dots \oplus N_r^{-1} \oplus \mathbb{I}_Y)) \\ &\cong \mathbb{P}(N_1^{-1} \oplus \dots \oplus N_r^{-1} \oplus \mathbb{I}_Y) \\ &= \mathbb{P}(N_{Y/X}^* \oplus \mathbb{I}_Y) = \tilde{X}. \end{aligned}$$

We also have that $\mathcal{O}_{\mathbb{P}(M_1 \oplus M_2 \oplus \dots \oplus M_{r+1})}(1) = \mathcal{O}_{\mathbb{P}(N_1^{-1} \oplus N_2^{-1} \oplus \dots \oplus N_r^{-1} \oplus \mathbb{I}_Y)}(1) \otimes \pi^* M_{r+1} = \tilde{L}$. \square

Note that, under this choice, we have $N_{Y/X} = \bigoplus_{\lambda=1}^r (L \otimes \pi^* M_\lambda^{-1})|_Y$. To use the maximal construction argument in §4, we have to use the following lemma.

LEMMA 3.3. In this situation, we have $SB(\tilde{L}) \subset \tilde{Y}$.

Proof. Recall that s_λ is the canonical section of the line bundle $[\Gamma_\lambda] = \pi^* M_\lambda^{-1} \otimes \tilde{L}$ (Lemma 2.1). For every global section $f \in H^0(Y, \pi^* M_\lambda^m)$ ($m \geq 0$), we have that $s_\lambda^m \otimes \pi^* f \in H^0(\tilde{X}, \tilde{L}^m)$. By the assumption of Theorem 1.2, the line bundle $L|_Y \otimes N_\lambda^{-1}$ is ample for each $\lambda = 1, \dots, r$. Therefore, for sufficiently large m , we can choose global sections of \tilde{L} whose common zero is Γ_λ . Using this argument for each λ , we have that $SB(\tilde{L}) \subset \tilde{Y}$. \square

3.3. Proof of Theorem 3.1.

3.3.1. *The outline of the proof.* We obtain (i) easily from the construction of u_α . We will prove (ii) and (iii) in §3.3.2. Now we shall explain the outline proof of (iv). Fix a Kähler form ω on Y . Define functions $\hat{V}^\mathbb{Q}$ and $\hat{V}_B^\mathbb{Q}$ on X by

$$\hat{V}^\mathbb{Q} := \sup_{\substack{\alpha = (\alpha_1, \dots, \alpha_r) \\ \alpha \in \square_L \cap \mathbb{Q}^r}}^* u_\alpha,$$

and

$$\hat{V}_B^\mathbb{Q} := \sup_{\substack{\alpha = (\alpha_1, \dots, \alpha_r) \\ \alpha \in \square_L \cap \mathbb{Q}^r}}^* \left[\sum_{\lambda=1}^{r+1} \alpha_\lambda \log |s_\lambda|_{\tilde{h}_\lambda}^2 + \pi^* V_{h_1^{\alpha_1} h_2^{\alpha_2} \dots h_{r+1}^{\alpha_{r+1}}, B} \right],$$

where $V_{h_1^{\alpha_1} h_2^{\alpha_2} \dots h_{r+1}^{\alpha_{r+1}}, B}$ is a function on Y defined as in Example 2.5 with respect to the volume form ω^n on Y . In §3.5, we prove that $V_{\theta_L, B} \leq \hat{V}_B^\mathbb{Q} + C$ holds for some constant C , where $V_{\theta_L, B}$ is also defined as in Example 2.5 (we specify the volume form on X later in §3.3.3). Then we have that, for some $C' \geq 0$,

$$V_{\theta_L, B} \leq \hat{V}_B^\mathbb{Q} + C \leq \hat{V}^\mathbb{Q} + C' \leq \hat{V} + C'.$$

Here, the second inequality follows from the equation (1) before the proof of Proposition 2.6.

3.3.2. *Proof of Theorem 3.1 (ii) and (iii).* In this subsection, we will show the upper semicontinuity of \widehat{V} . For simplicity of notation, we write V_α instead of $V_{\theta_\alpha} = \sup\{\psi \in PSH(Y, \theta_\alpha) \mid \psi \geq 0\}$. We will show the following proposition.

PROPOSITION 3.4. *The function $F: \square_L \times Y \rightarrow \mathbb{R} \cup \{\infty\} : F(\alpha, y) := V_\alpha(y)$ is u.s.c.*

From this proposition and compactness of \square_L , the standard argument shows (2) and (3) of Theorem 3.1. To prove Proposition 3.4, we need the following lemma.

LEMMA 3.5. *Let α and β be a point in \square_L .*

(i) *If $\alpha \leq \beta$ (i.e. $\alpha_\lambda \leq \beta_\lambda$ for each λ), $\frac{V_\alpha}{1-|\alpha|} \leq \frac{V_\beta}{1-|\beta|}$.*

(ii) *$\lim_{\beta \downarrow \alpha} \frac{V_\beta}{1-|\beta|} = \frac{V_\alpha}{1-|\alpha|}$, where $\lim_{\beta \downarrow \alpha}$ means the limit as β approaches to α under the condition $\alpha \leq \beta$ in (i).*

Proof. (i) Consider the local weights $\varphi_\alpha := \sum_{\lambda=1}^r \alpha_\lambda \varphi_\lambda + (1-|\alpha|)\varphi_{r+1}$. First, we use the equation

$$\varphi_\beta + \frac{1-|\beta|}{1-|\alpha|} \cdot V_\alpha = \frac{1-|\beta|}{1-|\alpha|} \cdot (\varphi_\alpha + V_\alpha) + \sum_{\lambda=1}^r \left(\beta_\lambda - \frac{1-|\beta|}{1-|\alpha|} \cdot \alpha_\lambda \right) \cdot \varphi_\lambda.$$

As the right-hand side is semi-positive, we have that $\frac{1-|\beta|}{1-|\alpha|} \cdot V_\alpha$ is θ_β -psh. Since this function is non-positive, we obtain

$$\frac{1-|\beta|}{1-|\alpha|} \cdot V_\alpha \leq V_\beta$$

by the definition of V_β .

(ii) Take a sequence $\{\beta^{(\nu)}\}_{\nu=1}^\infty$, $\beta^{(\nu)} = (\beta_1^{(\nu)}, \beta_2^{(\nu)}, \dots, \beta_r^{(\nu)})$, with $\beta_\lambda^{(\nu)} \downarrow \alpha_\lambda$ for each $\lambda = 1, 2, \dots, r$. We shall prove

$$\lim_{\nu \rightarrow \infty} \frac{V_{\beta^{(\nu)}}}{1-|\beta^{(\nu)}|} = \frac{V_\alpha}{1-|\alpha|}.$$

By (i), the inequality $\lim_{\nu \rightarrow \infty} \frac{V_{\beta^{(\nu)}}}{1-|\beta^{(\nu)}|} = \frac{V_\alpha}{1-|\alpha|}$ holds. Hence it is sufficient to prove the converse inequality.

Let us consider the local weight

$$\frac{1}{1-|\beta^{(\nu)}|} \cdot (\varphi_{\beta^{(\nu)}} + V_{\beta^{(\nu)}}) = \varphi_{r+1} + \sum_{\lambda=1}^r \frac{\beta_\lambda^{(\nu)}}{1-|\beta^{(\nu)}|} \cdot \varphi_\lambda + \frac{V_{\beta^{(\nu)}}}{1-|\beta^{(\nu)}|}.$$

By the right-hand side, this weight is clearly decreasing in ν . Moreover, by focusing on the left-hand side, we have that this weight is psh. Therefore the limit $\varphi_\alpha/(1-|\alpha|) + \lim_{\nu \rightarrow \infty} V_{\beta^{(\nu)}}/(1-|\beta^{(\nu)}|)$ is also psh. As the function $(1-|\alpha|) \cdot \lim_{\nu \rightarrow \infty} \frac{V_{\beta^{(\nu)}}}{1-|\beta^{(\nu)}|}$ is non-positive and θ_α -psh, we have that

$$(1-|\alpha|) \cdot \lim_{\nu \rightarrow \infty} \frac{V_{\beta^{(\nu)}}}{1-|\beta^{(\nu)}|} \leq V_\alpha.$$

□

Proof of Proposition 3.4. Fix a point $(\alpha^0, y^0) \in \square_L \times Y$. We shall prove the upper semi-continuity of F at (α^0, y^0) .

First, we treat the case where $|\alpha^0| = 1$. It is sufficient to prove

$$\limsup_{\square_L \times Y \ni (\alpha, y) \rightarrow (\alpha^0, y^0)} V_\alpha(y) \leq V_{\alpha^0}(y^0).$$

Since the forms $\theta_1, \theta_2, \dots, \theta_r$ are positive, θ_{α^0} is also positive. Thus $V_{\alpha^0}(y^0) = 0$ and upper semicontinuity is trivial in this case.

Next, we treat the case when $|\alpha^0| < 1$. In this case, we can take $\varepsilon > 0$ such that $\alpha^0 + \varepsilon := (\alpha_1^0 + \varepsilon, \alpha_2^0 + \varepsilon, \dots, \alpha^r + \varepsilon)$ lies in the interior of \square_L . Take a sufficiently small neighborhood $U_{\alpha^0, \varepsilon}$ in \square_L of α^0 such that every point α in $U_{\alpha^0, \varepsilon}$ satisfies $\alpha \leq \alpha^0 + \varepsilon$. It follows from Lemma 3.5 (i) that

$$\limsup_{\square_L \times Y \ni (\alpha, y) \rightarrow (\alpha^0, y^0)} \frac{V_\alpha(y)}{1 - |\alpha|} = \limsup_{U_{\alpha^0, \varepsilon} \times Y \ni (\alpha, y) \rightarrow (\alpha^0, y^0)} \frac{V_\alpha(y)}{1 - |\alpha|} \leq \limsup_{Y \ni y \rightarrow y^0} \frac{V_{\alpha^0 + \varepsilon}(y)}{1 - |\alpha^0 + \varepsilon|}$$

holds. By the upper semicontinuity of $\frac{V_{\alpha^0 + \varepsilon}}{1 - |\alpha^0 + \varepsilon|}$, we have

$$\limsup_{\square_L \times Y \ni (\alpha, y) \rightarrow (\alpha^0, y^0)} \frac{V_\alpha(y)}{1 - |\alpha|} \leq \frac{V_{\alpha^0 + \varepsilon}(y^0)}{1 - |\alpha^0 + \varepsilon|}.$$

Letting $\varepsilon \rightarrow 0$, we have

$$\limsup_{\square_L \times Y \ni (\alpha, y) \rightarrow (\alpha^0, y^0)} \frac{V_\alpha(y)}{1 - |\alpha|} \leq \frac{V_{\alpha^0}(y^0)}{1 - |\alpha^0|},$$

which shows the upper semicontinuity of the function $F(\alpha, y)/(1 - |y|)$ near (α^0, y^0) . By multiplying a continuous function $(\alpha, y) \mapsto 1 - |\alpha|$, we have that F itself is also u.s.c. \square

3.3.3. Integral formula. In the following, we consider the local coordinate open set $U_j^{(r+1)}$ defined as in §2. Recall that we defined a homogeneous fiber coordinate $[x_{j,1} : x_{j,2} : \dots : x_{j,r+1}]$ on $U_j^{(r+1)}$. We define the fiber coordinate z_1, \dots, z_r on $U_j^{(r+1)}$ as $z_\lambda := x_{j,\lambda}/x_{j,r+1}$. Then, we have $s_\lambda = z_\lambda$ ($1 \leq \lambda \leq r$) and $s_{r+1} = 1$. We can rewrite $\widehat{V}_B^{\mathbb{Q}}$ as follows:

$$\widehat{V}_B^{\mathbb{Q}} = \sup_{\substack{\ell_\lambda, m \in \mathbb{Z} \\ \ell/m \in \square_L}} \left[\frac{2\ell_1}{m} \log |z_1| + \dots + \frac{2\ell_r}{m} \log |z_r| + (\varphi_{\ell/m})_B(y) \right],$$

where $\varphi_{\ell/m}$ is defined by $(\ell_1/m)\varphi_1 + (\ell_2/m)\varphi_2 + \dots + (\ell_r/m)\varphi_r + (1 - (\ell_1 + \ell_2 + \dots + \ell_r)/m)\varphi_{r+1}$ and ϕ_B denotes a local weight corresponding to a function $V_{\phi, B}$. Note that we regard z_λ as a holomorphic function on $U_j^{(r+1)}$ here.

Define a volume form dV on X by setting

$$dV := \frac{\pi^* \omega^n}{n!} \wedge \frac{e^{\varphi_1 + \dots + \varphi_{r+1}}}{(|z_1|^2 e^{\varphi_1} + \dots + |z_r|^2 e^{\varphi_r} + e^{\varphi_{r+1}})^{r+1}} (\sqrt{-1})^r dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_r \wedge d\bar{z}_r$$

on $U_j^{(r+1)}$. Here, ω is a fixed Kähler form on Y as in §3.3.1. A simple calculation shows that this form is actually a global smooth volume form on X .

In the proof of Theorem 3.1, we need the following integral formula. Here we integrate a function $F = |z_1|^{2t_1} \dots |z_r|^{2t_r} e^{-\varphi_L} = \frac{|z_1|^{2t_1} \dots |z_r|^{2t_r}}{|z_1|^2 e^{\varphi_1} + \dots + |z_r|^2 e^{\varphi_r} + e^{\varphi_{r+1}}}$ by using the volume form dV .

LEMMA 3.6.

$$\begin{aligned} \int_{(z_1, \dots, z_r) \in \mathbb{C}^r} \frac{|z_1|^{2t_1} \dots |z_r|^{2t_r} \cdot e^{\varphi_1 + \dots + \varphi_{r+1}}}{(|z_1|^2 e^{\varphi_1} + \dots + |z_r|^2 e^{\varphi_r} + e^{\varphi_{r+1}})^{r+2}} (\sqrt{-1})^r dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_r \wedge d\bar{z}_r \\ = (2\pi)^r \frac{\Gamma(1+t_1) \dots \Gamma(1+t_r) \Gamma(2-(t_1+\dots+t_r))}{\Gamma(r+2) e^{\varphi_t}}. \end{aligned}$$

Proof. To make ideas clear, we first prove the case that $r = 2$. In this case, the equation we want to prove can be written as follows:

$$\begin{aligned} \int_{(z_1, z_2) \in \mathbb{C}^2} \frac{|z_1|^{2t_1} |z_2|^{2t_2} \cdot e^{\phi_1 + \phi_2 + \phi_3}}{(|z_1|^2 e^{\phi_1} + |z_2|^2 e^{\phi_2} + e^{\phi_3})^4} (\sqrt{-1})^2 dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \\ = (2\pi)^2 \frac{\Gamma(1+t_1) \Gamma(1+t_2) \Gamma(2-(t_1+t_2))}{\Gamma(4) e^{\phi_t}}, \end{aligned}$$

where $\phi_t := t_1 \phi_1 + t_2 \phi_2 + (1 - (t_1 + t_2)) \phi_3$. Write $z_1 = a_1 e^{i\theta_1}$ and $z_2 = a_2 e^{i\theta_2}$ and define A_λ by $A_\lambda := e^{\phi_\lambda}$ ($\lambda = 1, 2, 3$). Then we have

$$\begin{aligned} \int_{\mathbb{C}^2} \frac{|z_1|^{2t_1} |z_2|^{2t_2} \cdot A_1 A_2 A_3}{(|z_1|^2 A_1 + |z_2|^2 A_2 + A_3)^4} (\sqrt{-1})^2 dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \\ = (2\pi)^2 2^2 \int_{\mathbb{R}_{>0}^2} \frac{a_1^{1+2t_1} a_2^{1+2t_2} A_1 A_2 A_3}{(a_1^2 A_1 + a_2^2 A_2 + A_3)^4} da_1 da_2. \end{aligned}$$

Next, we use the following polar coordinate (s, θ) :

$$\begin{aligned} s &:= \left(\frac{A_1}{A_3} a_1^2 + \frac{A_2}{A_3} a_2^2 \right)^{1/2} \in \mathbb{R}_{\geq 0}, \\ \theta &:= \text{Arctan} \left(\frac{\sqrt{A_2} a_2}{\sqrt{A_1} a_1} \right) \in [0, \pi/2]. \end{aligned}$$

Note that this coordinate can be written as $a_1 = \sqrt{A_3/A_1} \cdot s \cos \theta$ and $a_2 = \sqrt{A_3/A_2} \cdot s \sin \theta$. Then we have

$$\begin{aligned} \int_{\mathbb{R}_{>0}^2} \frac{a_1^{1+2t_1} a_2^{1+2t_2} A_1 A_2 A_3}{(a_1^2 A_1 + a_2^2 A_2 + A_3)^4} da_1 da_2 \\ = A_1^{-t_1} A_2^{-t_2} A_3^{t_1+t_2-1} \int_{s=0}^{\infty} \frac{s^{3+2t_1+2t_2}}{(s^2+1)^4} ds \int_{\theta=0}^{\pi/2} (\cos \theta)^{1+2t_1} (\sin \theta)^{1+2t_2} d\theta. \end{aligned}$$

We denote this value by I . We will calculate the integration in s and θ respectively.

First we consider the integration in s . To compute, we use the substitution $\sigma = s^2$. Then,

$$\int_{s=0}^{\infty} \frac{s^{3+2t_1+2t_2}}{(s^2+1)^4} ds = \frac{1}{2} \int_{\sigma=0}^{\infty} \frac{\sigma^{1+t_1+t_2}}{(\sigma+1)^4} d\sigma = \frac{1}{2} B(2+t_1+t_2, 2-t_1-t_2).$$

At the last equality, we use the formula [OLBC, 5.12.3] for the beta function.

Next, we consider the integration in θ . By the formula [OLBC, 5.12.2], we have

$$\int_{\theta=0}^{\pi/2} (\cos \theta)^{1+2t_1} (\sin \theta)^{1+2t_2} d\theta = \frac{1}{2} B(1+t_1, 1+t_2).$$

By [OLBC, 5.12.1], we have

$$(2\pi)^2 2^2 I = (2\pi)^2 A_1^{-t_1} A_2^{-t_2} A_3^{t_1+t_2-1} \frac{\Gamma(2-t_1-t_2)\Gamma(1+t_1)\Gamma(1+t_2)}{\Gamma(4)}.$$

The proof in the case that $r = 2$ is complete.

Now we will prove the theorem in the general case. Since the proof is almost the same, we only explain the essential points. We use the coordinate change $z_\lambda = a_\lambda e^{i\theta_\lambda}$ and get the expression of a_1, \dots, a_r . Then we use the r -dimensional polar coordinate

$$\begin{aligned} a_1 &= \sqrt{A_{r+1}/A_1} \cdot s \cos \theta_1 \\ a_2 &= \sqrt{A_{r+1}/A_2} \cdot s \sin \theta_1 \cos \theta_2 \\ a_3 &= \sqrt{A_{r+1}/A_3} \cdot s \sin \theta_1 \sin \theta_2 \cos \theta_3 \\ &\vdots \\ a_{r-1} &= \sqrt{A_{r+1}/A_{r-1}} \cdot s \sin \theta_1 \sin \theta_2 \sin \theta_3 \cdots \sin \theta_{r-2} \cos \theta_{r-1} \\ a_r &= \sqrt{A_{r+1}/A_r} \cdot s \sin \theta_1 \sin \theta_2 \sin \theta_3 \cdots \sin \theta_{r-2} \sin \theta_{r-1}, \end{aligned}$$

where $A_\lambda = e^{\varphi_\lambda}$. The determinant of the Jacobian matrix is written as

$$\sqrt{\frac{A_r}{A_1} \cdots \frac{A_r}{A_{r-1}}} s^{r-2} (\sin \theta_1)^{r-3} (\sin \theta_2)^{r-4} \cdots (\sin \theta_{r-3})^1.$$

Finally we can use formulae of the beta function to deduce the conclusion. \square

3.3.4. Proof of Theorem 3.1 (iv). As we described in §3.3.1, we shall prove $V_{\theta_L, B} \leq \widehat{V}_B^{\mathbb{Q}} + C$. Fix a global section $F \in H^0(X, L^m)$ with

$$\int_X |F|_{h_L^m}^2 dV = 1.$$

We will decompose F into orthogonal components using the following claim.

CLAIM 3.7. *The direct decomposition of $H^0(X, L^m)$ induced by the isomorphisms $H^0(X, L^m) = H^0(X, \mathcal{O}_X(m)) = H^0(Y, S^m(M_1 \otimes \cdots \otimes M_{r+1})) \cong \bigoplus_{\ell_1 + \cdots + \ell_{r+1} = m} H^0(Y, M_1^{\ell_1} \otimes \cdots \otimes M_{r+1}^{\ell_{r+1}})$ is the orthogonal decomposition with respect to the L^2 -norm defined by Hermitian metric h_L^m of L^m and the volume form dV . Here, $S^m E$ denotes the m -th symmetric tensor of E .*

Proof. By the decomposition above, we have injective morphisms $H^0(Y, M_1^{\ell_1} \otimes \cdots \otimes M_{r+1}^{\ell_{r+1}}) \rightarrow H^0(X, L^m)$ for each $(r+1)$ -tuple of non-negative integers $\ell = (\ell_1, \dots, \ell_{r+1})$ with $\ell_1 + \cdots + \ell_{r+1} = m$. In the following, $M_1^{\ell_1} \otimes \cdots \otimes M_{r+1}^{\ell_{r+1}}$ will be denoted by M^ℓ . This morphism maps $f_\ell \in H^0(Y, M^\ell)$ to $s_1^{\ell_1} \cdot s_2^{\ell_2} \cdots s_{r+1}^{\ell_{r+1}} \pi^* f_\ell$.

We will prove that, for each $\ell = (\ell_1, \dots, \ell_{r+1})$ and $\ell' = (\ell'_1, \dots, \ell'_{r+1})$, two sections $\beta := s_1^{\ell_1} \cdot s_2^{\ell_2} \cdots s_{r+1}^{\ell_{r+1}} \pi^* f_\ell$ and $\beta' := s_1^{\ell'_1} \cdot s_2^{\ell'_2} \cdots s_{r+1}^{\ell'_{r+1}} \pi^* f_{\ell'}$ are orthogonal if $\ell \neq \ell'$. By using the local trivialization, we regard β and β' as holomorphic functions. Then, by the equations $s_\lambda = z_\lambda$ ($\lambda = 1, 2, \dots, r$) and $s_{r+1} = 1$,

$$\begin{aligned} \int_X \langle \beta, \beta' \rangle_{h_L^m} dV &= \int_X (z_1^{\ell_1} \cdot z_2^{\ell_2} \cdots z_r^{\ell_r} \pi^* f_\ell) \cdot \overline{(z_1^{\ell'_1} \cdot z_2^{\ell'_2} \cdots z_r^{\ell'_r} \pi^* f_{\ell'})} e^{-m\varphi_L} dV \\ &= \int_{y \in Y} \left[\int_{z \in \pi^{-1}(y)} \frac{(z_1^{\ell_1} \cdots z_r^{\ell_r} \pi^* f_\ell) \overline{(z_1^{\ell'_1} \cdots z_r^{\ell'_r} \pi^* f_{\ell'})} e^{-m\varphi_L} e^{\varphi_1 + \cdots + \varphi_{r+1}}}{(|z_1|^2 e^{\varphi_1} + \cdots + |z_r|^2 e^{\varphi_r} + e^{\varphi_{r+1}})^{r+1}} (\sqrt{-1})^r dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_r \wedge d\bar{z}_r \right] \frac{\omega^n}{n!}. \end{aligned}$$

Write $z_\lambda = s_\lambda r^{i\theta_\lambda}$. Considering integration in θ_λ 's, we have that it becomes 0 if $\ell \neq \ell'$. Therefore, two sections are orthogonal for different ℓ and ℓ' . \square

Let us decompose $F \in H^0(X, L^m)$ into the sum of the components $\beta_\ell = s_1^{\ell_1} \cdot s_2^{\ell_2} \cdots s_{r+1}^{\ell_{r+1}} \pi^* f_\ell$, where $f_\ell \in H^0(Y, M_1^{\ell_1} \otimes \cdots \otimes M_{r+1}^{\ell_{r+1}})$, according to the orthogonal decomposition obtained by the claim. By the orthogonality, we have

$$\int_X |\beta_\ell|_{h_L^m}^2 dV \leq \int_X |F|_{h_L^m}^2 dV (= 1).$$

Next we will estimate the norm of f_ℓ .

CLAIM 3.8. *There exist constants C_1 and C_2 independent of m such that*

$$\int_{y \in Y} |f_\ell(y)|^2 e^{-m(\varphi_{\ell/m})} \frac{\omega^n}{n!} \leq C_1 C_2^m \int_X |\beta_\ell|^2 e^{-m\varphi_L} dV,$$

where $\varphi_{\ell/m}$ stands for $\frac{\ell_1}{m}\varphi_1 + \frac{\ell_2}{m}\varphi_2 + \cdots + \frac{\ell_r}{m}\varphi_r + (1 - \frac{\ell_1 + \cdots + \ell_r}{m})\varphi_{r+1}$ as in §3.3.3.

Proof. Let $z^\ell := z_1^{\ell_1} \cdot z_2^{\ell_2} \cdots z_r^{\ell_r}$. Then we can write as $z^\ell \pi^* f_\ell = \beta_\ell$ under the trivialization. We will estimate the right-hand side from below. We have that

$$\begin{aligned} \int_X |\beta_\ell|^2 e^{-m\varphi_L} dV &= \int_X |z^\ell f_\ell(x)|^2 e^{-m\varphi_L} dV \\ &= \int_{y \in Y} |f_\ell(x)|^2 \left[\int_{z \in \pi^{-1}(y)} \frac{|z^\ell|^2 e^{-m\varphi_L} e^{\varphi_1 + \cdots + \varphi_{r+1}}}{(|z_1|^2 e^{\varphi_1} + \cdots + |z_r|^2 e^{\varphi_r} + e^{\varphi_{r+1}})^{r+1}} (\sqrt{-1})^r dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_r \wedge d\bar{z}_r \right] \frac{\omega^n}{n!}. \end{aligned}$$

Let us denote by dP the measure on the fiber $\pi^{-1}(y)$ defined as

$$dP := \frac{e^{\varphi_1 + \cdots + \varphi_{r+1}}}{(|z_1|^2 e^{\varphi_1} + \cdots + |z_r|^2 e^{\varphi_r} + e^{\varphi_{r+1}})^{r+1}} (\sqrt{-1})^r dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_r \wedge d\bar{z}_r.$$

Then by Hölder's inequality, we have that

$$\int_{z \in \pi^{-1}(y)} |z^\ell|^{2/m} \cdot e^{-\varphi_L} dP \leq \left[\int_{\pi^{-1}(y)} |z^\ell|^2 \cdot e^{-m\varphi_L} dP \right]^{\frac{1}{m}} \cdot \left[\int_{\pi^{-1}(y)} 1 \cdot dP \right]^{\frac{m-1}{m}}.$$

A straightforward computation similar to the proof of Lemma 3.6 shows that the value of the integral $\int_{\pi^{-1}(y)} dP$ in the right-hand side is a constant independent of y , which we

will denote by I . By Lemma 3.6, the integral in the left-hand side is equal to

$$(2\pi)^r \frac{\Gamma\left(1 + \frac{\ell_1}{m}\right) \cdots \Gamma\left(1 + \frac{\ell_r}{m}\right) \Gamma\left(2 - \left(\frac{\ell_1 + \cdots + \ell_r}{m}\right)\right)}{\Gamma(r+2)e^{\varphi_\ell}}.$$

As $\Gamma(t)$ is bounded for $1 \leq t \leq 2$ from below, there exists a positive constant $C > 0$ such that $\Gamma(t) \geq C$ for $1 \leq t \leq 2$. Combining these estimates, we have that

$$(2\pi)^r \frac{C^{r+1}e^{-\varphi_\ell}}{\Gamma(r+2)} \leq \left[\int_{z \in \pi^{-1}(y)} |z^\ell|^2 e^{-m\varphi_L} dP \right]^{\frac{1}{m}} \cdot I^{\frac{m-1}{m}}.$$

Thus we obtain

$$\begin{aligned} \int_X |\beta_\ell|^2 e^{-m\varphi_L} dV &= \int_{y \in Y} |f_\ell(y)|^2 \left[\int_{z \in \pi^{-1}(y)} |z^\ell|^2 e^{-m\varphi_L} dP \right] \frac{\omega^n}{n!} \\ &\geq \int_{y \in Y} |f_\ell(y)|^2 I^{-(m-1)} \left[(2\pi)^r \frac{C^{r+1}e^{-\varphi_\ell}}{\Gamma(r+2)} \right]^m \frac{\omega^n}{n!} \\ &= \frac{(2\pi)^{rm} C^{rm} I^{-(m-1)}}{\Gamma(r+2)^m} \int_{y \in Y} |f_\ell(y)|^2 e^{-m\varphi_\ell} \frac{\omega^n}{n!}, \end{aligned}$$

and the claim is proved. \square

By the previous estimates and the definition of the Bergman-type metric, we have

$$\frac{1}{m} \log |f_\ell|^2 \leq (\varphi_\ell)_B + \frac{\log C_1 + m \log C_2}{m}.$$

To prove the desired inequality $V_{\theta_L, B} \leq \widehat{V}_B^{\mathbb{Q}} + C$, we will estimate the norm of the section $F = \sum_\ell z^\ell \pi^* f_\ell$ from above. Assume $\ell = (\ell_1, \dots, \ell_r)$ satisfies $\ell/m \in \square_L$. By the argument as in the proof of [K2, Proposition 2.5(2)], we have

$$\frac{2}{m} \log |F| \leq \max_\ell \left[\frac{2\ell_1}{m} \log |z_1| + \cdots + \frac{2\ell_r}{m} \log |z_r| + (\phi_\ell/m)_B \right] + \frac{1}{m} \log C_1 + \log C_2.$$

Let $C := \log C_1 + \log C_2$. Then the right-hand side is estimated by $\widehat{V}_B^{\mathbb{Q}} + C$ by definition. Since we can take m arbitrarily, the supremum of the left-hand side over m and F is $V_{\theta_L, B}$, and the proof is complete.

4. PROOF OF THE MAIN RESULTS

In this section, we prove Theorem 1.1 and Theorem 1.2. Let X, Y , and L be those as in Theorem 1.2 (Especially, we assume that Y admits a holomorphic tubular neighborhood). The idea of the proof is based on [K2]: we first construct a new “projective bundle model” $(\tilde{X}, \tilde{Y}, \tilde{L})$ from (X, Y, L) , and construct a minimal singular metric of L by using the metric on \tilde{L} as in §3. See also §3.2 for the relation between the models (X, Y, L) and $(\tilde{X}, \tilde{Y}, \tilde{L})$.

4.1. The projective bundle model $(\tilde{X}, \tilde{Y}, \tilde{L})$. Let X, Y , and L be as in Theorem 1.2. Denote by \tilde{X} the total space of the projective bundle $\mathbb{P}(\mathbb{I}_Y \oplus N_{Y/X}^*)$, \tilde{Y} the subvariety $\mathbb{P}(\mathbb{I}_Y)$, and by \tilde{L} the line bundle $\mathcal{O}_{\mathbb{P}(\mathbb{I}_Y \oplus N_{Y/X}^*)}(1) \otimes \pi^* L|_Y$, where $\pi: \tilde{X} \rightarrow Y$ is the natural projection. Note that $\tilde{X} = \mathbf{P}(\mathbb{I}_Y \oplus N_{Y/X})$ and $\tilde{Y} = \mathbf{P}(\mathbb{I}_Y)$. It is easily observed that \tilde{X} is a compactified space of the normal bundle $N_{Y/X}$, and from this point of view, \tilde{Y} can be regarded as a zero section of $N_{Y/X}$. Therefore, by the assumption on the existence of a holomorphic tubular neighborhood, we can take a neighborhood V of Y in X and \tilde{V} of \tilde{Y} in \tilde{X} such that there exists a biholomorphic map $i: \tilde{V} \cong V$ with $i|_{\tilde{V}} = \pi|_{\tilde{V}}$.

PROPOSITION 4.1. *$i^* L \cong \tilde{L}$ holds by shrinking V and \tilde{V} if necessary.*

The proof is based on [K2, §3]. First we prove the following proposition as a higher codimensional analogue of [K2, Proposition 3.1]:

PROPOSITION 4.2.

By shrinking V suitably, the following holds:

- (1) *(a version of Rossi's theorem) The natural map $H^1(V, \mathcal{O}_V) \rightarrow H^1(V, \mathcal{O}_V/I_Y^n)$ is injective for some $n \geq 1$, where I_Y the defining ideal sheaf of $Y \subset V$.*
- (2) *If $H^1(Y, S^\ell N_{Y/X}^*)$ vanishes for each $\ell \geq 1$, then the groups $\text{Pic}^0(V)$ and $\text{Pic}^0(Y)$ are isomorphic.*

Proof of Proposition 4.2. (1) See the proof of [K2, Proposition 3.1] (We intrinsically use Rossi's theorem [R, Theorem 3]. Here we remark that, by Lemma 4.3 below, we may assume that V is a strongly pseudoconvex domain which has Y as a maximal compact set). \square

LEMMA 4.3. *There exists a strongly pseudoconvex holomorphic tubular neighborhood V of Y which has Y as a maximal compact analytic set.*

Proof. As Y admits a holomorphic tubular neighborhood, it is sufficient to show the lemma by assuming $X = \tilde{X}$. Take a C^∞ Hermitian metric $h_{N_\lambda^{-1}}$ on N_λ^{-1} with positive curvature. Take a local coordinate y of Y and, by pulling it by π , we regard $(z, y) = (z_1, z_2, \dots, z_r, y)$ as a local coordinates system of X , where z_λ is a fiber coordinate of N_λ . By considering the sublevel set of the function $\Phi: N_{Y/X} \rightarrow \mathbb{R}$ defined by

$$\Phi(z, y) := \sum_{\lambda=1}^r |z_\lambda|^2 e^{\varphi_\lambda(y)},$$

the lemma is shown (φ_λ is the local weight of $h_{N_\lambda^{-1}}$). \square

Proof of Proposition 4.1. First note that

$$K_Y^{-1} \otimes S^\ell N_{Y/X}^* = \bigoplus_{\alpha \in \mathbb{Z}_{\geq 0}^r, |\alpha|=\ell} K_Y^{-1} \otimes \left(\bigotimes_{\lambda=1}^r N_\lambda^{-\alpha_\lambda} \right)$$

holds. As $K_Y^{-1} \otimes N_\lambda^{-1}$ and N_μ are positive for each $\lambda, \mu = 1, 2, \dots, r$, it follows from Kodaira's vanishing theorem that $H^1(Y, S^\ell N_{Y/X}^*)$ vanishes for each $\ell \geq 1$. Thus, by Proposition 4.2, it is sufficient to show that $\tilde{L}|_{\tilde{Y}} \cong \pi|_{\tilde{Y}}^* L$ holds. The line bundle $\mathcal{O}_{\mathbb{P}(\mathbb{I}_Y \oplus N_{Y/X}^*)}(1)$

corresponds to the divisor $\mathbb{P}(N_{Y/X}^*) \subset \mathbb{P}(\mathbb{I}_Y \oplus N_{Y/X}^*)$, which does not intersect \tilde{Y} . Therefore we have $\tilde{L}|_{\tilde{Y}} = \mathcal{O}_{\mathbb{P}(\mathbb{I}_Y \oplus N_{Y/X}^*)}(1)|_{\tilde{Y}} \otimes \pi|_{\tilde{Y}}^* L = \pi|_{\tilde{Y}}^* L$. \square

4.2. Minimal singular metrics on L and \tilde{L} . Let X, Y , and L be as in Theorem 1.2, and $\tilde{X}, \tilde{Y}, \tilde{L}, V$, and \tilde{V} be as in the previous subsection. Here we prove the following:

PROPOSITION 4.4. *Let h be a minimal singular metric of L and \tilde{h} be a minimal singular metric of \tilde{L} . Then $h|_V \sim_{\text{sing}} (i^{-1})^* \tilde{h}|_{\tilde{V}}$ holds at each point in V .*

The proof of Proposition 4.4 is based on the “maximum construction technique” which is also used in the proof of [K2, Theorem 1.2]. Fix a C^∞ metric h_∞ of L and denote by θ the curvature tensor Θ_{h_∞} . Set $\varphi_V := \sup\{\varphi \in PSH(V, \theta|_V) \mid \varphi \leq 0 \text{ on } V\}$ and $\varphi_X := \sup\{\varphi \in PSH(X, \theta) \mid \varphi \leq 0 \text{ on } X\}$. We first show the following:

LEMMA 4.5. $\varphi_V \sim_{\text{sing}} \varphi_X$. *Especially, the restriction of a minimal singular metric of L to V has the equivalent singularity as the metric $h_\infty|_V \cdot e^{-\varphi_V}$.*

Proof. As the inequality $\varphi_X|_V \leq \varphi_V$ is easily obtained, all we have to do is to show the existence of a constant C with $\varphi_V \leq \varphi_X|_V + C$. As L is big and $\text{SB}(L) = Y$, we can take an integer $m \geq 1$ and sections $f_1, f_2, \dots, f_\ell \in H^0(X, L^m)$ such that the common zero of these sections is Y [Laz, 2.1.21]. Denote by h_a the Bergman type metric on L constructed from f_1, f_2, \dots, f_ℓ . Define the function φ_a by $h_a = h_\infty \cdot e^{-\varphi_a}$. We may assume that $\varphi_a \leq 0$ holds on V . Fix a relatively compact open neighborhood $V_0 \Subset V$ of Y and set $C_1 := -\inf_{V \setminus V_0} \varphi_a$. Consider a θ -psh function $\hat{\varphi} := \max\{\varphi_V - C_1, \varphi_a\}$. As $\varphi_V - C_1 \leq -C_1 \leq \varphi_a$ holds on $V \setminus V_0$, we have $\hat{\varphi} = \varphi_a$ on each point in $V \setminus V_0$. Thus we can extend $\hat{\varphi}$ to whole X by defining $\hat{\varphi}(x) := \varphi_a(x)$ for each $x \in X \setminus V_0$. It is clear from the construction that $\hat{\varphi} \in PSH(X, \theta)$. Set $C_2 := \max_X \hat{\varphi}$. Then, as $\hat{\varphi} - C_2 \leq 0$, we obtain that $\hat{\varphi} - C_2 \leq \varphi_X$. Therefore it holds that $\varphi_V - C_1 - C_2 \leq \hat{\varphi} - C_2 \leq \varphi_X$. \square

Proof of Proposition 4.4. Take a C^∞ Hermitian metric h_∞ on L and \tilde{h}_∞ on \tilde{L} with $h_\infty|_V = i^* \tilde{h}_\infty$ (here we used Proposition 4.1). By Lemma 4.5, it follows that $\varphi_V \sim_{\text{sing}} \varphi_X$, where φ_V and φ_X are as above. Set $\varphi_{\tilde{V}} := \sup\{\varphi \in PSH(\tilde{V}, \Theta_{\tilde{h}_\infty}|_{\tilde{V}}) \mid \varphi \leq 0 \text{ on } \tilde{V}\}$ and $\varphi_{\tilde{X}} := \sup\{\varphi \in PSH(\tilde{X}, \Theta_{\tilde{h}_\infty}) \mid \varphi \leq 0 \text{ on } \tilde{X}\}$. By the arguments in §3.2, we can apply Lemma 4.5 also to the projective bundle model (\tilde{X}, \tilde{L}) to obtain that $\varphi_{\tilde{V}} \sim_{\text{sing}} \varphi_{\tilde{X}}$. As $h_\infty|_V = i^* \tilde{h}_\infty$, we have that $\varphi_V = i^* \varphi_{\tilde{V}}$. Therefore it follows that $\varphi_X \sim_{\text{sing}} i^* \varphi_{\tilde{X}}$. \square

4.3. Proof of Theorem 1.2. Theorem 1.2 follows from Theorem 3.1 and Proposition 4.4. \square

5. A SUFFICIENT CONDITION FOR THE EXISTENCE OF A HOLOMORPHIC TUBULAR NEIGHBORHOOD AND PROOF OF THEOREM 1.1

Let X be a complex manifold and $Y \subset X$ be a compact complex submanifold of codimension r . In this section, we investigate when does Y admit a holomorphic tubular neighborhood V in X . Especially, we here study a higher codimensional analogue of Grauert’s theorem ([G], the case of $r = 1$, see also Theorem 5.4 below).

5.1. A higher codimensional analogue of Grauert's theorem. In this subsection, we show the following:

PROPOSITION 5.1. *Assume that $N_{Y/X}$ admits a direct decomposition $N_{Y/X} = N_1 \oplus N_2 \oplus \cdots \oplus N_r$ into r negative line bundles. Assume also that $H^1(E, \mathcal{O}_{\mathbf{P}(N_{Y/X})}(\nu)) = 0$ and $H^1(E, T_E \otimes \mathcal{O}_{\mathbf{P}(N_{Y/X})}(\nu)) = 0$ hold for each $\nu \geq 1$, where we denote by E the total space of the projective bundle $\mathbf{P}(N_{Y/X})$. Then Y admits a holomorphic tubular neighborhood.*

Note that Proposition 5.1 is the Grauert's theorem when $r = 1$. We will consider a blow-up $p: W \rightarrow X$ of X along Y and apply the Grauert's theorem to $E \subset W$ to show this proposition, where we are regarding E as the exceptional divisor (see [D1, Proposition 12.4]). For this purpose, we first show the following:

LEMMA 5.2. *Assume that E admits a holomorphic tubular neighborhood in W . Then Y also admits a holomorphic tubular neighborhood in X .*

Proof. Denote by \tilde{Y} the zero section of $\pi: N_{Y/X} \rightarrow X$. Take a neighborhood \tilde{V} of \tilde{Y} . We denote by \tilde{W} the blow-up $\tilde{p}: \tilde{W} \rightarrow \tilde{V}$ of \tilde{V} along \tilde{Y} and by $\tilde{E} \subset \tilde{W}$ the exceptional set. By the assumption (and by shrinking X if necessary), we may assume that there exists a biholomorphic map $F: \tilde{W} \rightarrow W$ with $F|_{\tilde{E}} = q|_{\tilde{E}}$, where $q: \mathcal{O}_{\mathbf{P}(N_{Y/X})}(-1) \rightarrow E (= \mathbf{P}(N_{Y/X}))$ is the natural projection (here we are regarding \tilde{W} as a neighborhood of the zero section E of the line bundle $\mathcal{O}_{\mathbf{P}(N_{Y/X})}(-1)$, see also [D1, Proposition 12.4]).

First, let us construct a holomorphic map $g: V \rightarrow Y$ with $g|_Y = \text{id}_Y$ such that the diagram

$$\begin{array}{ccccc} \tilde{W} & \xrightarrow{F} & W & \xrightarrow{p} & V \\ \downarrow & \circlearrowleft & \downarrow & \circlearrowleft & \downarrow g \\ \tilde{E} & \xrightarrow{q|_{\tilde{E}}} & E & \longrightarrow & Y \end{array}$$

is commutative. It is clear that such function g is uniquely determined in the set-theoretic sense. It is also followed from a standard argument that this g is a continuous map. It is also clear that $g|_{V \setminus Y}$ is biholomorphic and that $g|_Y = \text{id}_Y$. Thus the existence of the holomorphic map g follows from Riemann's extension theorem (see Lemma 5.3 below).

Next we show that the biholomorphic map $f: \tilde{V} \setminus \tilde{Y} \cong V \setminus Y$ induced by $F|_{\tilde{W} \setminus \tilde{E}}$ extends to the biholomorphism $\hat{f}: \tilde{V} \cong V$ with $\hat{f}|_{\tilde{Y}} = \pi|_{\tilde{Y}}$. Note that, by construction, the fibration structures $\pi|_{\tilde{V}}: \tilde{V} \rightarrow \tilde{Y}$ and $g: V \rightarrow Y$ are preserved by f . Therefore, by a simple topological observation, it follows that there uniquely exists a continuous map $\hat{f}: \tilde{V} \rightarrow V$ with $\hat{f}|_{\tilde{V} \setminus \tilde{Y}} = f$, and that this \hat{f} satisfies $\hat{f}|_{\tilde{Y}} = \pi|_{\tilde{Y}}$. The regularity of \hat{f} and \hat{f}^{-1} is shown again by Lemma 5.3 below. \square

LEMMA 5.3. *Let M and N be complex manifolds, $Z \subset M$ a submanifold with codimension greater than or equal to 1, and $h: M \rightarrow N$ be a continuous map. Assume that $h|_{M \setminus Z}$ is holomorphic. Then h is holomorphic on M .*

Proof. Take a point $z \in Z$ and a coordinate open ball $U' \subset N$ with coordinates system $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ around $h(z)$ ($n := \dim N$). We may assume that $U' = \{|\eta| < \varepsilon\}$ for

some $\varepsilon > 0$. Take a sufficiently small neighborhood $U \subset M$ of z so that $U \subset h^{-1}(U')$ (and thus $h(U) \subset U'$). In what follows, we show the lemma by replacing M with U , N with U' , and h with $h|_U$ (Note that here we are regarding N as an open ball of \mathbb{C}^n , especially). It is sufficient to show each function h_λ is holomorphic on z , where $h = (h_1, h_2, \dots, h_n)$ is the decomposition by the coordinates system $\eta = (\eta_1, \eta_2, \dots, \eta_n)$. As h_λ is continuous (and thus it is locally bounded), we may assume that the L^2 -norm of $h_\lambda|_{U \setminus Z}$ is bounded by shrinking U if necessary. Therefore it follows from Riemann's extension theorem that we can extend $h_\lambda|_{U \setminus Z}$ to a holomorphic function $\hat{h}_\lambda: U \rightarrow \mathbb{C}$. Since $U \setminus Z \subset U$ is a dense subset, we can conclude that $h_\lambda = \hat{h}_\lambda$, which proves the lemma. \square

By Lemma 5.2, all we have to do is to investigate when does E admit a holomorphic tubular neighborhood in W . We apply the following Grauert's theorem to this problem:

THEOREM 5.4 ([G, Satz 7, p. 363], see also [CM, Theorem 4.4]). *Let M be a complex manifold, $Z \subset M$ be a strongly exceptional subvariety of pure codimension 1. Assume that $H^1(Z, N_{Z/M}^{-\nu}) = 0$ and $H^1(Z, T_Z \otimes N_{Z/M}^{-\nu}) = 0$ hold for each $\nu \geq 1$. Then Z has a holomorphic tubular neighborhood in M .*

Proof of Proposition 5.1. By the assumption, Lemma 5.2 and Theorem 5.4, all we have to do is to show that $E \subset W$ is an exceptional subset (in the sense of Grauert). By [Lau, Theorem 4.9, 6.12], [G, Satz 8, p. 353], and [HR, Lemma 11] (see also [CM, Theorem 3.6]), it is sufficient to see the following two conditions: (i) $N_{E/W}$ is negative, and (ii) $\mathcal{O}_W/I_E^2 \cong \mathcal{O}_{\tilde{W}}/I_{\tilde{E}}^2$, where $I_E \subset \mathcal{O}_W$ and $I_{\tilde{E}} \subset \mathcal{O}_{\tilde{W}}$ are the defining ideal sheaves of E and \tilde{E} , respectively. (i) follows from $N_{E/W}^{-1} = \mathcal{O}_{\mathbf{P}(N_{Y/X})}(1) = \mathcal{O}_{\mathbb{P}(N_1^{-1} \oplus N_2^{-1} \oplus \dots \oplus N_r^{-1})}(1)$ and the assumption that each N_λ is negative. (ii) follows from the condition $H^1(E, T_E \otimes \mathcal{O}_{\mathbf{P}(N_{Y/X})}(1)) = 0$ (see [CM, Proposition 1.10, 1.11]). \square

5.2. A sufficient condition for the existence of a holomorphic tubular neighborhood. In this subsection, we show the following lemma as an application of Proposition 5.1:

LEMMA 5.5. *Let X be a complex manifold and Y be a compact complex submanifold. Assume that $N_{Y/X}$ admits a direct decomposition $N_{Y/X} = N_1 \oplus N_2 \oplus \dots \oplus N_r$ into r negative line bundles. Assume also the following three conditions: (i) $N_\lambda \cong N_\mu$ for each λ and μ , (ii) $N_\lambda^{-1} \otimes K_Y^{-1} \otimes T_Y$ is Nakano positive, and (iii) $N_\lambda^{-1} \otimes K_Y^{-1}$ is ample for each λ . Then Y admits a holomorphic tubular neighborhood in X .*

Note that, when T_Y is holomorphically trivial, the conditions (ii) and (iii) are automatically satisfied.

Proof. By Proposition 5.1, it is sufficient to show $H^1(E, \mathcal{O}_{\mathbf{P}(N_{Y/X})}(\nu)) = 0$ and $H^1(E, T_E \otimes \mathcal{O}_{\mathbf{P}(N_{Y/X})}(\nu)) = 0$ for each $\nu \geq 1$, where $E := \mathbf{P}(N_{Y/X})$. Note that it follows from the assumption (i) that $E \cong Y \times \mathbf{P}^r$. By the relative Euler sequence

$$0 \rightarrow \mathbb{I}_E \rightarrow p|_E^* N_{Y/X} \otimes \mathcal{O}_{\mathbf{P}(N_{Y/X})}(1) \rightarrow T_{E/Y} \rightarrow 0,$$

it turns out that it is sufficient to show the following four vanishing assertions: $H^1(E, \mathcal{O}_{\mathbf{P}(N_{Y/X})}(\nu)) = 0$, $H^2(E, \mathcal{O}_{\mathbf{P}(N_{Y/X})}(\nu)) = 0$, $H^1(E, \mathcal{O}_{\mathbf{P}(N_{Y/X})}(\nu+1) \otimes p|_E^* N_\lambda) = 0$, and $H^1(E, \mathcal{O}_{\mathbf{P}(N_{Y/X})}(\nu) \otimes$

$p|_E^* T_Y) = 0$ for each $\nu \geq 1$. By Nakano's vanishing theorem, the problem is reduced to show Nakano positivity for the following three vector bundles: $K_E^{-1} \otimes \mathcal{O}_{\mathbf{P}(N_{Y/X})}(1)$, $K_E^{-1} \otimes \mathcal{O}_{\mathbf{P}(N_{Y/X})}(2) \otimes p|_E^* N_\lambda$, and $K_E^{-1} \otimes \mathcal{O}_{\mathbf{P}(N_{Y/X})}(1) \otimes p|_E^* T_Y$. As

$$K_E^{-1} \cong p|_E^*(N^r \otimes K_Y^{-1}) \otimes \mathcal{O}_{\mathbf{P}(N_{Y/X})}(r)$$

holds ($N := N_1 \cong N_2 \cong \dots \cong N_r$), we can rewrite these three bundles by $\mathcal{O}_{\mathbf{P}(N_{Y/X})}(r+1) \otimes p|_E^*(N^r \otimes K_Y^{-1})$, $\mathcal{O}_{\mathbf{P}(N_{Y/X})}(r+2) \otimes p|_E^*(N^{r+1} \otimes K_Y^{-1})$, and $\mathcal{O}_{\mathbf{P}(N_{Y/X})}(r+1) \otimes p|_E^*(T_Y \otimes N^r \otimes K_Y^{-1})$, respectively. In what follows, we show the Nakano positivity for these three bundles.

First let us note that $\mathcal{O}_{\mathbf{P}(N_{Y/X})}(1) \otimes p|_E^* N = \text{Pr}_1^* \mathcal{O}_{\mathbf{P}^r}(1)$, where Pr_1 is the first projection $E \cong \mathbf{P}^r \times Y \rightarrow \mathbf{P}^r$. Let us denote by h the metric on this line bundle which is the pull-back of the Fubini-Study metric by Pr_1 . By tensoring h and metrics on $N^{-1} \otimes K_Y^{-1}$ and $T_Y \otimes N^{-1} \otimes K_Y^{-1}$ with Nakano positive curvature, we can show the Nakano positivity for the bundles $\mathcal{O}_{\mathbf{P}(N_{Y/X})}(r+1) \otimes p|_E^*(N^r \otimes K_Y^{-1}) = \mathcal{O}_{\mathbf{P}(\mathbb{I}_Y)}(r+1) \otimes p|_E^*(N^{-1} \otimes K_Y^{-1})$ and $\mathcal{O}_{\mathbf{P}(N_{Y/X})}(r+2) \otimes p|_E^*(N^{r+1} \otimes K_Y^{-1}) = \mathcal{O}_{\mathbf{P}(\mathbb{I}_Y)}(r+2) \otimes p|_E^*(N^{-1} \otimes K_Y^{-1})$.

Finally we show the Nakano positivity for $F := \mathcal{O}_{\mathbf{P}(N_{Y/X})}(r+1) \otimes p|_E^*(T_Y \otimes N^r \otimes K_Y^{-1}) = \mathcal{O}_{\mathbf{P}(\mathbb{I}_Y)}(r+1) \otimes p|_E^*(T_Y \otimes N^{-1} \otimes K_Y^{-1})$. Take a metric h' on $T_Y \otimes N^{-1} \otimes K_Y^{-1}$ with Nakano positive curvature. Then $h^{r+1} \otimes p|_E^* h'$ is a metric on F , whose curvature is $(r+1)\Theta_h \otimes \text{Id}_F + p|_E^* \Theta_{h'}$, which is easily seen to be Nakano positive. \square

5.3. Proof of Theorem 1.1. By Lemma 5.5, Y admits a holomorphic tubular neighborhood in X .

By Theorem 1.2, there exists a minimal singular metric $h_{\min, L}$ whose local weight $\varphi_{\min, L}$ can be written in the form

$$\varphi_{\min, L}(z, y) = \log \max_{\alpha \in \square_L} |z^\alpha|^2 e^{(\varphi_\alpha)_e(y)} + O(1).$$

By choosing metrics $e^{\varphi_\alpha(y)}$ as in [K1, §2.2], we obtain that $(\varphi_\alpha)_e(y) = \varphi_\alpha(y)$ holds and $\varphi_\alpha(y)$ depends continuously on (y, α) , which proves the assertion. \square

6. EXAMPLES

6.1. Nakayama's example. Nakayama's example (X, L, Y) is the example which admits no Zariski decomposition even after modifications [N, IV, §2.6] (see also [K1, §1]). In this example, the manifold X is a total space of the projective bundle $\pi: X := \mathbb{P}(M_1 \oplus M_2 \oplus M_3) \rightarrow Y$ over an abelian surface Y , where M_1 and M_2 are ample line bundles on Y and M_3 is a line bundle on Y . The line bundle L is the inverse of the tautological line bundle: i.e. $L := \mathcal{O}_{\mathbb{P}(M_1 \oplus M_2 \oplus M_3)}(1)$. Then the stable base locus of L is the subset $\mathbb{P}(M_3) \subset X$, which here we are regarding as Y . As it is clearly observed, this example (X, L, Y) is a special case of those in §3. Therefore we can apply Theorem 3.1 to this example. Especially, by using the metrics as in the proof of Theorem 1.1, we can reprove the main result in [K1].

6.2. Zariski's example and its higher (co-)dimensional analogues and proof of Corollary 1.4. In [K2, §4.2], we applied its main result (=Theorem 1.1, 1.2 for $r = 1$) to Zariski's and Mumford's example (X, L, Y) , in which L is nef and big however not semi-ample and showed the semi-positivity of L (i.e. the existence of a C^∞ Hermitian metric on L with semi-positive curvature). Here we construct an example which can be regarded as a higher-codimensional analogue of Zariski's example and apply Theorem 1.1 to it. In what follows, we only consider the case of $r = 2$ for simplicity.

Take two general quadric surfaces Q_1 and Q_2 in \mathbb{P}^3 . Then we may assume that the intersection $C := Q_1 \cap Q_2$ is a smooth elliptic curve and Q_1 and Q_2 intersects transversally along C . Fix N points p_1, p_2, \dots, p_N in C . Denote by $\pi: X := \text{Bl}_{\{p_1, p_2, \dots, p_N\}} \mathbb{P}^3 \rightarrow \mathbb{P}^3$ the blow-up of \mathbb{P}^3 at these N points, by Y the strict transform of C , by D_1 and D_2 the the strict transform of Q_1 and Q_2 , respectively, by E_λ the exceptional divisor $\pi^{-1}(p_\lambda)$ for each λ , by E the divisor $\sum_{\lambda=1}^N E_\lambda$, and by H the pull-back $\pi^* \mathcal{O}_{\mathbb{P}^3}(1)$. Note that $D_\lambda \in |2H - E|$.

Let us consider the line bundle $L := \mathcal{O}_X(H + D_1) = \mathcal{O}_X(3H - E)$ on X . As H is big and E is effective, L is also big. It is also observed that $\text{Bs}|L| \subset Y$ holds, since H is base point free and $\text{Bs}|H| \subset Y$ by construction. As the simple computation shows that the intersection number $(L.Y)$ is equal to $12 - N$, we can conclude that L is nef if and only if $12 \geq N$.

First let us consider the case of $N = 12$. In this case, we may assume that $L|_Y$ is an general (and thus non-torsion) element of $\text{Pic}^0(Y)$ by choosing p_1, p_2, \dots, p_{12} generally. Then it is easily observed that $\text{SB}(L) = Y$ holds, and therefore that L is not semi-ample, however L is nef and big. In this sense, we can regard this example (X, Y, L) as an analogue of Zariski's example with $r = 2$. As D_1 and D_2 intersects transversally along Y , we obtain the decomposition

$$N_{Y/X} = N_{D_1/X}|_Y \oplus N_{D_2/X}|_Y = \mathcal{O}_X(D_1)|_{D_1}|_Y \oplus \mathcal{O}_X(D_2)|_{D_2}|_Y = \mathcal{O}_X(D_1)|_Y \oplus \mathcal{O}_X(D_2)|_Y$$

By denoting $N_\lambda := \mathcal{O}_X(D_\lambda)|_Y$ for each $\lambda = 1, 2$, it holds that $N_1 \cong N_2$, $(D_\lambda.Y) = 2(H.Y) - (E.Y) = 8 - 12 < 0$, and $\deg_Y L|_Y \otimes N_\lambda^{-1} = 0 - (8 - 12) > 0$. Therefore we can apply Theorem 1.1, Corollary 1.3, and also Lemma 5.5 to our (X, L, Y) . By Corollary 1.3, we have that $h_{\min, L}|_Y$ is bounded, where $h_{\min, L}$ is a minimal singular metric of L (here we use that fact that $L|_Y$ admits a C^∞ Hermitian metric with zero curvature, since $L|_Y$ is a flat line bundle). Therefore we can conclude that $h_{\min, L}$ is bounded. Note that we can moreover show that the semi-positivity of L (i.e. that we can choose $h_{\min, L}$ as a C^∞ Hermitian metric) by applying Lemma 5.5 and use the “regularized maximum construction” technique as in [K2, Corollary 3.4].

Next let us consider the case of $N > 12$. In this case, L is not nef. By the argument as above, we also have $\text{SB}(L) = Y$, $\deg N_\lambda = 8 - N < 0$, and $\deg(L|_Y \otimes N_\lambda^{-1}) = (12 - N) - (8 - N) = 4 > 0$. Thus we can apply Theorem 1.1 to (X, Y, L) also in this case. As we can compute to obtain that

$$\square_L = \left\{ \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}_{\geq 0}^2 \left| \frac{N-12}{N-8} \leq |\alpha| \leq 1 \right. \right\},$$

it follows from Theorem 1.1 that the local weight function $\varphi_{\min,L}$ of a minimal singular metric $h_{\min,L}$ can be written as

$$\varphi_{\min,L}(z, y) = \log \max_{\alpha \in \square_L} \prod_{\lambda=1}^r |z_\lambda|^{2\alpha_\lambda} + O(1) = \frac{N-12}{N-8} \cdot \log(|z_1|^2 + |z_2|^2) + O(1)$$

on a neighborhood of each point of Y , where y is a coordinate of Y and $z = (z_1, z_2, \dots, z_r)$ is a system of local defining functions of Y . Especially in this case, $\varphi_{\min,L}$ has analytic singularities along Y .

Note that similar example can be constructed in general dimension by considering, for example, some points blow-up of a del Pezzo manifold of degree 1 (See [F] for example. For the choice of the counterpart of the divisors D_ν 's above, see [K3, §6.3]).

6.3. [BEGZ]'s example. The above two examples satisfies the assumptions (ii) and (iii) in §1. On the other hand, the example (X, Y, L) in [BEGZ, Example 5.4] does not satisfies these examples. A minimal singular metric of this L is unbounded and actually has singularities along Y , however the Lelong number of the local weight is 0 for each point in X (see also [K2, Example 4.2]). Especially, the conclusion of Theorem 1.2 does not hold in this example.

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